Practice exam — Analysis (WPMA14004)

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (4 + 6 points)

(a) State the Archimedean Property of $\mathbb{R}$.

(b) Use the Archimedean Property to prove that $\sup\left\{\dfrac{n-1}{n} \,:\, n \in \mathbb{N}\right\} = 1$.

Problem 2 (4 + 4 + 4 + 4 + 4 points)

(a) Let $\alpha > -1$. Prove by induction that $(1+\alpha)^n \geq 1 + n\alpha$ for all $n \in \mathbb{N}$.

(b) Let $y_n = \left(1 + \dfrac{1}{n}\right)^{n+1}$. Show that

$$\frac{y_{n-1}}{y_n} = \left(1 + \frac{1}{n^2-1}\right)^n \cdot \frac{n}{n+1} \quad \text{for all } n \geq 2.$$

(c) Use part (a) and (b) to prove that $y_{n-1} > y_n$ for all $n \geq 2$. Hint: α is allowed to depend on n.

(d) Prove that $\lim y_n$ exists.

(e) Let $x_n = \left(1 + \dfrac{1}{n}\right)^n$. Prove that $\lim x_n = \lim y_n$.

Problem 3 (15 points)

Let $A \subset \mathbb{R}$ be nonempty and both open and closed. Prove that A is unbounded.

Problem 4 (10 + 5 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$:

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

(a) Prove that f is uniformly continuous on $\mathbb{R}$. Hint: first consider $f : [0, 2T] \to \mathbb{R}$.

(b) Assume in addition that f is differentiable. Prove that $f'(x) = 0$ for infinitely many points $x \in \mathbb{R}$.

Problem 5 (6 + 9 points)

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 1$ for all $x \in \mathbb{R}$. Define the sequence

$$f_n(x) = \frac{nf(x)}{1 + nf(x)}.$$

(a) Compute the pointwise limit of (f_n).

(b) Does (f_n) converge uniformly on $\mathbb{R}$?

Problem 6 (6 + 9 points)

Define $h : [0, 2] \to \mathbb{R}$ as

$$h(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

(a) Prove that h is integrable on $[0, 2]$.

(b) Let $H(x) = \int_0^x h(t)dt$ for all $x \in [0, 2]$. Compute $H'(1)$.

End of test (90 points)

Solution of Problem 1 (4 + 6 points)

(a) Archimedean Property:

(i) For all $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$.
(2 points)

(ii) For all $y > 0$ there exists $n \in \mathbb{N}$ such that $1/n < y$.
(2 points)

(b) Clearly, $(n-1)/n < 1$ for all $n \in \mathbb{N}$, which implies that $s = 1$ is an upper bound for the given set.
(3 points)

Let $\epsilon > 0$ be arbitrary. By the Archimedean Property there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$. Therefore,

$$\frac{n-1}{n} = 1 - \frac{1}{n} > 1 - \epsilon = s - \epsilon.$$

This shows that $s - \epsilon$ is *not* an upper bound for the given set, and hence we conclude that $s = 1$ is the *least* upper bound.
(3 points)

Solution of Problem 2 (4 + 4 + 4 + 4 + 4 points)

(a) For $n = 1$ the inequality reads as $1 + \alpha \geq 1 + \alpha$, which is indeed true.
(1 point)

Now assume that $(1+\alpha)^n \geq 1 + n\alpha$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}(1+\alpha)^{n+1} &= (1+\alpha)^n(1+\alpha) \\ &\geq (1+n\alpha)(1+\alpha) \\ &= 1 + (n+1)\alpha + n\alpha^2 \\ &\geq 1 + (n+1)\alpha,\end{aligned}$$

which shows that the inequality holds for $n+1$ as well. By induction, the inequality holds for all $n \in \mathbb{N}$.
(3 points)

(b) Straightforward computations show that

$$\frac{y_{n-1}}{y_n} = \frac{\left(\dfrac{n}{n-1}\right)^n}{\left(\dfrac{n+1}{n}\right)^n\left(\dfrac{n+1}{n}\right)} = \left(\frac{n^2}{n^2-1}\right)^n \cdot \frac{n}{n+1} = \left(1 + \frac{1}{n^2-1}\right)^n \cdot \frac{n}{n+1}.$$

(4 points)

(c) Applying part (a) with $\alpha = 1/(n^2-1)$ to the result of part (b) gives

$$\frac{y_{n-1}}{y_n} \geq \left(1 + \frac{n}{n^2-1}\right) \cdot \frac{n}{n+1} > \left(1 + \frac{1}{n}\right) \cdot \frac{n}{n+1} = 1 \quad \Rightarrow \quad y_{n-1} > y_n.$$

(4 points)

(d) From part (c) it follows that (y_n) is a decreasing sequence. Also observe that $y_n > 0$ for all $n \in \mathbb{N}$.
(2 points)

The Monotone Convergence Theorem states that a decreasing sequence that is bounded below converges. We conclude that $y = \lim y_n$ exists.
(2 points)

(e) Note that

$$x_n = y_n \cdot \frac{n}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Hence, by the Algebraic Limit Theorem it follows that

$$\lim x_n = \lim y_n \cdot \lim \frac{n}{n+1} = \lim y_n.$$

(4 points)

Solution of Problem 3 (15 points)

This problem has two solutions, which, in fact, are quite similar.

Solution 1. Assume that A is bounded, then $s = \sup A$ exists. Now we need to force a contradiction.
(4 points)

Note that $s \in \overline{A}$ (the closure of A). Since A is closed we have $A = \overline{A}$ and therefore it follows that $s \in A$.
(4 points)

Since A is open there exists $\delta > 0$ such that $V_\delta(s) \subset A$. Therefore there exists $x \in A$ with $x > s$. (For example take $x = s + \frac{1}{2}\delta$.)
(4 points)

But as $s = \sup A$ it follows that $x \leq s$ for all $x \in A$. This gives a contradiction, and we conclude that A is unbounded.
(3 points)

Remark: one can give a similar proof using the infimum.

Solution 2. Assume that A is bounded, then A is compact since it was already assumed that A is closed.
(4 points)

The function $f(x) = x$ is continuous on A. By compactness of A the function f attains a maximum on A, i.e., there exists a point $x_0 \in A$ such that $f(x_0) \geq f(x)$ for all $x \in A$.
(4 points)

Since A is open there exists $\delta > 0$ such that $V_\delta(x_0) \subset A$. Therefore, there exists $x_1 \in A$ with $x_1 > x_0$ so that $f(x_1) > f(x_0)$.
(4 points)

This contradicts the fact that f attains its maximum in x_0. Hence, we conclude that A must be unbounded.
(3 points)

Remark: one can give a similar proof using the minimum of f.

Solution of Problem 4 (10 + 5 points)

(a) The interval $[0, 2T]$ is closed and bounded and therefore compact. Since the function f is continuous it is uniformly continuous on $[0, 2T]$.
(2 points)

For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in [0, 2T].$$

(2 points)

Assume that $\delta < T$ (otherwise we just make δ smaller). Now let $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta$. There exists $k \in \mathbb{Z}$ such that

$$x + kT, y + kT \in [0, 2T].$$

(2 points)

Using the fact that f is T-periodic gives

$$\begin{aligned} |x - y| < \delta \ &\Rightarrow \ |(x + kT) - (y + kT)| < \delta \\ &\Rightarrow \ |f(x + kT) - f(y + kT)| < \epsilon \\ &\Rightarrow \ |f(x) - f(y)| < \epsilon \end{aligned}$$

which proves that f is uniformly continuous on $\mathbb{R}$.
(4 points)

(b) Let $k \in \mathbb{Z}$ be arbitrary. Note that $f(kT) = f((k+1)T)$ by the fact that f is T-periodic. Since f is continuous on $[kT, (k + 1)T]$ and differentiable on $(kT, (k + 1)T)$ Rolle's Theorem implies that there exists a point $c_k \in (kT, (k + 1)T)$ such that $f'(c_k) = 0$. Hence, $f'(x) = 0$ for infinitely many points $x \in \mathbb{R}$.
(5 points)

Solution of Problem 5 (6 + 9 points)

(a) Let $x_0 \in \mathbb{R}$ be arbitrary, then by the Algebraic Limit Theorem it follows that

$$\lim f_n(x_0) = \lim \frac{nf(x_0)}{1 + nf(x_0)} = \lim \frac{f(x_0)}{f(x_0) + 1/n} = \frac{f(x_0)}{f(x_0) + \lim(1/n)} = \frac{f(x_0)}{f(x_0)} = 1.$$

We conclude that (f_n) converges pointwise to the constant function $f(x) = 1$.
(6 points)

(b) Since $f(x) \geq 1$ for all $x \in \mathbb{R}$ it follows that

$$|f_n(x) - 1| = \frac{1}{1 + nf(x)} \leq \frac{1}{1 + n} \qquad \text{for all } x \in \mathbb{R}.$$

(4 points)

Therefore, it follows that

$$\sup_{x \in \mathbb{R}} |f_n(x) - 1| \leq \frac{1}{1 + n} \quad \text{so that} \quad \lim \left(\sup_{x \in \mathbb{R}} |f_n(x) - 1| \right) = 0.$$

We conclude that (f_n) converges to $f(x) = 1$ uniformly on $\mathbb{R}$.
(5 points)

Solution of Problem 6 (6 + 9 points)

(a) Let $\epsilon > 0$ be arbitrary and take the partition

$$P_\epsilon = \{0, 1 - \tfrac{1}{4}\epsilon, 1 + \tfrac{1}{4}\epsilon, 2\}.$$

With this partition it easily follows that

$$M_1 = 1, \quad M_2 = 1, \quad M_3 = 1, \quad m_1 = 1, \quad m_2 = 0, \quad m_3 = 1.$$

(3 points) Therefore,

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{3} (M_k - m_k)(x_k - x_{k-1}) = (M_2 - m_2)(x_2 - x_1) = \tfrac{1}{2}\epsilon < \epsilon.$$

We conclude that h is integrable on $[0, 2]$.
(3 points)

(b) Define $H(x) = \int_0^x h(t)dt$. This makes sense as we have already shown that h is integrable on $[0, 2]$ and therefore also on each subinterval $[0, x] \subset [0, 2]$. Note that we *cannot* apply the Fundamental Theorem of Calculus to say that $H'(1) = h(1) = 0$ since h is not continuous at $x = 1$!

Define the function

$$g(x) = 1 - h(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

By using the same partition as in part (a) it follows that

$$U(f, P_\epsilon) = \tfrac{1}{2}\epsilon, \qquad L(f, P_\epsilon) = 0.$$

This shows that g is also integrable on $[0, 2]$ and $\int_0^2 g = 0$.
(4 points)

In particular, it follows that $\int_0^x g = 0$ for all $x \in [0, 2]$. Therefore,

$$H(x) = \int_0^x h = \int_0^x (h - 1 + 1) = \int_0^x (1 - g) = \int_0^x 1 - \int_0^x g = \int_0^x 1 = x$$

for all $x \in [0, 2]$.
(4 points)

Therefore, $H'(x) = 1$ for all $x \in [0, 2]$. In particular it follows that $H'(1) = 1 \neq 0$.
(1 point)